Table 1 Accuracy of the integration rule for $I(0)$

$f(\eta)$	Exact	Two-point	Four-point
e^η	−0.9716595	−0.9719117	−0.9716595
$\cos^{-1}\eta$	−3.1415926	−3.1415926	−3.1415926
$\sqrt{1+\eta^2}$	−1.0656799	−1.0717967	−1.0657672
$(1-\eta^2)\log(1+\eta^2)$	0.2240980	0.383576	0.2259250

5) There is no numerical difficulty involved in obtaining the weights and abscissas because the standard Gauss-Legendre coefficients used can be found to great precision from tables or stable numerical computations.[6]

The effectiveness of this quadrature rule is demonstrated in Table 1, where $⨍_{-1}^{1} f(\eta)/\eta^2 d\eta$ for four different functions $f(\eta)$ is shown. The exact value is compared with two and four-point rules and it can be seen that the convergence is quite satisfactory.

Finite value integrals can be scaled and translated. A further useful property, which facilitates the computation of $I(y)$ for general limits of integration, is that

$$⨍_a^b \frac{f(\eta)}{(\eta-y)^2} d\eta = \int_a^{y-c} \frac{f(\eta)}{(\eta-y)^2} d\eta + ⨍_{y-c}^{y+c} \frac{f(\eta)}{(\eta-y)^2} + \int_{y+c}^{b} \frac{f(\eta)}{(\eta-y)^2} d\eta \tag{8}$$

with $y+c$ and $y-c$ within the limits of the original integral. This statement can be proved rigorously.[7] The finite part can then be evaluated using Eq. (7) after the appropriate scaling and translation, although care should be taken to obtain the nonsingular parts for $c \ll 1$ because of the large $1/\eta^2$ contribution. A special Gauss rule with $1/y^2$ weighting may be used in this case.

Conclusion

A special quadrature rule, which can be used for the direct numerical evaluation of the finite part of a double-pole singular integral, has been presented. Its application to subsonic aerodynamic problems greatly simplifies the algebra involved in lifting surface theories, a fact that makes it possible to use the kernel function approach for complex problems. The author is currently involved in applying this technique to compute time-accurate transient aerodynamic loads on wings.[10]

To date, no attempt has been made to investigate with full generality the convergence properties of the given quadrature rule.

Acknowledgments

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References

[1]Bisplinghoff, R. L. and Ashley, H., *Principles of Aeroelasticity*, Dover Publications, New York, 1962, Chap. 4.

[2]Mangler, K. W., "Improper Integrals in Theoretical Aerodynamics," British Royal Aeronautical Establishment, Rept. Aero 2424, 1951.

[3]Rowe, W. S., Winther, B.A., and Redman, M. C., "Prediction of Unsteady Aerodynamic Loadings Caused by Trailing Edge Control Surface Motions in Subsonic Compressible Flow—Analysis and Results," NASA CR 2003, 1976.

[4]Monegato, G., Istuto Matematico, Politecnico di Torino, Italy, private communications, 1985. See also *Computing*, Vol. 29, pp. 355-359 for related work.

[5]Hildebrand, F. B., *Introduction to Numerical Analysis*, 2nd ed., McGraw-Hill Book Co., New York, 1974.

[6]Stroud, A. H., *Numerical Quadrature and Solution of Ordinary Differential Equations*, Springer-Verlag, New York, 1974, Chap. 8.

[7]Kutt, H. R., "On the Numerical Evaluation of Finite Part Integrals Involving an Algebraic Singularity," National Research Institute for Mathematical Sciences, Pretoria, CSIR Special Rept. WISK 179, 1975, p. 65 (available from National Technical Information Service as N77-10835 and N77-10837).

[8]Chawla, M. M. and Kumar, S., "Convergence of Quadratures for Cauchy Principal Value Integrals," *Computing*, Vol. 23, 1979, pp. 67-72.

[9]Ioakimidis, N. I., "On the Numerical Evaluation of Derivatives of Cauchy Principal Value Integrals," *Computing*, Vol. 27, 1981, pp. 81-88.

[10]van Niekerk, B., "A Rational Approach to Lifting Surface Theory with Application to High Angles of Attack," Ph.D. Thesis, Department of Aeronautics and Astronautics, Stanford, CA, June 1986.

Quasi-One-Dimensional Gas/Particle Nozzle Flows with Shock

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Nomenclature

A	= cross-sectional area, m^2
A_g	= cross-sectional area occupied by the gas phase, m^2
A_p	= total surface of the particles, m^2
b	= velocity of sound in gas/particle mixture, m/s
c	= velocity of sound in gas phase, m/s
c_D	= particle drag coefficient
c_p, c_v	= gas specific heat at constant pressure and volume, respectively, J/kg K
c_s	= particle heat capacity, J/kg K
d	= particle diameter, m
E_g, E_p	= internal energy per unit volume, J/m^3
f	= drag coefficient ratio
F	= interaction force between the phases
G_g, G_p	= mass flow, kg/s
h	= convective heat transfer coefficient, W/m^{2}k
H_g	= total enthalpy, J/kg
k	= thermal conductivity (gas), W/mk
M, M_p	= gas- and particle-phase Mach numbers, respectively
N	= number of particles per unit volume
Nu	= particle Nusselt number
p	= gas-phase pressure, N/m^2
Q	= heat flux per unit length, W/m^3
R	= gas constant, J/kg K
Re_p	= particle Reynolds number
S	= slip ratio
T_g, T_p	= temperature, K
T_r	= temperature ratio T_p/T_g
u_g, u_p	= velocities, m/s
x	= axial coordinate, m
α, α_p	= void fraction
β	= loading ratio, G_p/G_g
μ_g	= gas viscosity, Ns/m^2
κ	= specific heat ratio
ρ_g, ρ_p	= density, kg/m^3

Superscripts

(*)	= bulk conditions

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Subscripts

g,p = gas- and particle-phase, respectively

Introduction

GAS/PARTICLE flows have been studied from several points of view. Many authors have contributed to nozzle flows; the earlier theories (see Rudinger[1,2] and Kriebel[3]), were based on the quasi-gas approximations, i.e., the properties of gas were modified due to the presence of the particles. The advantage of these theories is that classical-gas dynamic theories can be used with the properties of the quasi-gas. These theories have been developed to include slip between the gas and particles in velocity and temperature (Kliegel[4]). From about 1960, theories based on the consideration that the particle phase is a continuum have been presented, in particular, by Wallis,[5] Di Giacinto et al.,[6] Chang,[7] and Sharma and Crowe.[8] Chang and Di Giacinto presented two-dimensional time-marching steady-state solutions of gas/particle flows, where two sets of mass, momentum, and energy equations have been solved simultaneously. Recently, Chang[9] published a three-dimensional steady-state calculation of gas/particle nozzle flows.

Rudinger,[10] Kriebel,[3] and Wallis[5] presented methods for solving normal shock waves in gas/particle suspensions. These methods are strongly limited to one-dimensional flows and have a disadvantage in quasi-one-dimensional flow, e.g., in a convergent/divergent nozzle. The disadvantage is due to the location of the shock; to avoid this problem the time-dependent equations can be used.

In this Note, a time-dependent method is applied to the solution of the fully-coupled gas/particle, quasi-one-dimensional flow inside nozzles. The time-dependent method is shown to yield good resolution of the entire flow region where we have subsonic, transonic, and supersonic flow including shock.

Characteristic Parameters and Governing Equations

It is possible to choose different physical models to describe the gas/particle flow in correspondence to different conditions and values of the characteristic parameters which appear in the governing equations. Parameters in connection with the physical condition are considered first.

Let the solid particles be spherical with diameter d, constant material density ρ_p, and the number density N of the particles sufficiently large to consider the solid phase as a continuum. That is, a volume of the order of $\mathcal{O}(\delta^3)$ with $d \ll \delta \ll L$, where L is the characteristic length scale of the flowfield.

The gas-phase void fraction is

$$\alpha = A_g/A = [1/(1+\beta\rho_g u_g/\rho_p u_p)] \tag{1}$$

where β is the loading ratio of the flow and defined as the ratio between the mass flow of the solid and gas phase, and u_g, u_p the gas and particle velocities, respectively.

In the present study the values of α are in the order of 99% and hence the particle phase void fraction

$$\alpha_p = 1-\alpha \tag{2}$$

is in the order of 1%. These small values of α_p indicate a dilute suspension and, hence, the pressure in the solid phase may be neglected.

If both the gas and the solid phase are a continuum, a set of mass, momentum, and energy equations can be written for each phase of adiabatic, quasi-one-dimensional nozzle flows.

For the gas phase, the equations become
Continuity:

$$\frac{\partial \rho_g^*}{\partial t} + \frac{1}{A}\frac{\partial G_g}{\partial x} = 0 \tag{3}$$

Momentum:

$$\frac{\partial G_g}{\partial t} + \frac{\partial}{\partial x}(G_g u_g) = -A\frac{\partial p}{\partial x} - F \tag{4}$$

Energy:

$$\frac{\partial E_g}{\partial t} + \frac{1}{A}\frac{\partial}{\partial x}(G_g H_g) = -Q - F(u_g - u_p) \tag{5}$$

where $\rho_g^* = \alpha\rho_g$ is the gas phase bulk density, $G_g = \rho_g^* u_g A$ the gas mass flow, $E_g = \rho_g^*(c_v T_g + \frac{1}{2}u_g^2)$ the internal energy per unit volume, and $H_g = c_p T_g + \frac{1}{2}u_g^2$ the total enthalpy; F is the interaction force between the phases and Q the heat transfer between the phases.

For the particle phases, the equations become
Continuity:

$$\frac{\partial \rho_p^*}{\partial t} + \frac{1}{A}\frac{\partial G_p}{\partial x} = 0 \tag{6}$$

Momentum:

$$\frac{\partial G_p}{\partial t} + \frac{\partial}{\partial x}(G_p u_p) = F \tag{7}$$

Energy:

$$\frac{\partial E_p}{\partial t} + \frac{1}{A}\frac{\partial}{\partial x}(A u_p E_p) = Q + F(u_g - 2u_p) \tag{8}$$

where $\rho_p^* = (1-\alpha)\rho_p$ is the particle phase bulk density, $G_p = \rho_p^* u_p A$ the particle mass flow, and $E_p = \rho_p^* c_s T_g$ the internal energy per unit volume; c_s is the heat capacity for the solid particles.

The two sets of equations are coupled through the interaction force F (or the momentum interchange force), the void fraction α, and the heat transfer.

The interaction force F, approximately given by the sum of the drag force applied on each of the N particles and per unit volume of the suspension, becomes

$$F = 18\alpha_p \mu_g (f/d^2)(u_g - u_p) \tag{9}$$

where μ_g is the dynamic viscosity of the gas phase and the drag coefficient ratio f is[1]

$$f = c_D/c_{D\,\text{Stokes}} = 1 + (1/6)Re_p^{2/3} \tag{10}$$

where Re_p is the particle Reynolds number.

The spherical particles with the temperature T_p are assumed to have uniform temperature within the particle. The heat transfer between the phases (per unit length) is then

$$Q = 6A(\alpha_p Nuk/d^2)(T_g - T_p) \tag{11}$$

The Nusselt number Nu is dependent on the particle Reynolds and Prandtl numbers, in this case, a relation given by Carlson and Høglund[11] has been applied.

The problem contains the seven unknowns α, ρ_g, u_g, u_p, p, T_g and T_p, the six equations [Eqs. (3-8)], and the equation of state for the gas phase, which will close the problem.

The time-dependent equations [Eqs. (3-8)], which are hyperbolic, can then be used to obtain the steady-state solution for the quasi-one-dimensional nozzle flows. The advantage of the time-dependent equations is that it is easy to take care of singularities as shocks and for gas Mach number $M=1$.

The gas Mach number is defined as

$$M = u_g / c \tag{12}$$

where $c^2 = (\partial p / \partial \rho)_{s=\text{const}} = \kappa R T_g$.

The particle Mach number is furthermore defined as

$$M_p = u_p / c \tag{13}$$

Numerical Procedure

Integration of Eqs. (3-8) through time to reach a steady-state solution requires a proper numerical scheme. A very large number of schemes have been developed, both explicit and implicit. Implicit schemes inevitably involve some form of matrix inversion but allow much larger time steps to be taken. For simplicity, an explicit scheme for gas/particle flow calculation will be used herein.

Explicit schemes are limited to time steps that satisfy the Courant-Friedrichs-Lewy (CFL) condition, i.e.,

$$\Delta t = \Delta x / (u_g + c) \tag{14}$$

A Lax-Wendroff-Richtmyer (LWR) scheme has been used to integrate Eqs. (3-8). This scheme is a two-step method which has second-order accuracy in time and space. The disadvantage of scheme with second-order accuracy is that they give oscillation around discontinuities as shock. To prevent some of this oscillation, artificial or numerical viscosity can be employed. In order to solve the shock condition in the nozzle, artificial viscosity is introduced by adding a small part of upwind differencing to the LWR scheme, i.e.,

$$\text{new value} = (1-\theta)\text{LWR} + \theta\text{upwind} \tag{15}$$

where $0 < \theta < 0.2$. The discontinuities occur only in the gas phase, and the artificial viscosity is only applied to the equations at this phase. For the particle phase, the pressure term is omitted and hence the discontinuities. With supersonic downstream condition, all the variables are calculated in order to satisfy the downstream moving characteristics. For subsonic downstream condition one of the characteristics is moving upstream and one of the variables has to be given. In this case, the static pressure downstream is given and the momentum equation removed.

Application to Nozzle Flow

The physical model described in this Note has been applied to the flow inside the Jet Propulsion Laboratory (JPL) axisymmetric nozzle.[7,12] (See Fig. 1.) Figure 2 shows the location and the strength of the shock in the JPL nozzle for loading ratios 0, 0.176, 0.429, and 1, with $d = 2\mu$m. The figure shows that the strength of the frozen shock decreases with increased loading, and the location moves upstream with increased loading. This is caused by the upstream condition of the shock, the pressure rise due to the frozen shock, the relaxation between the two phases behind the frozen shock, and the change of the cross-sectional area, in order to satisfy a given back pressure. Furthermore, Fig. 2 shows an interesting result for $\beta = 1$. The frozen shock does not occur and the gas-phase supersonic condition is not obtained. Figure 3 shows a supersonic downstream solution for the same condition (mentioned above), and shows that this condition includes a shock. This shock can be expressed as a mixture shock. To be sure that this is a shocked flow solution for the two-phase nozzle flow, the condition can be checked by a nonequilibrium quasi-gas calculation. The velocity of sound in a nonequilibrium gas/particle suspension is given as[13]

$$\left(\frac{b}{c}\right)^2 = \frac{[1+\beta(c_s/c_p)T_r]}{\alpha^2(1+\beta S)[1+\kappa\beta(c_s/c_p)T_r] + \alpha\beta(\kappa-1)S(1-S)} \tag{16}$$

where $S = u_p/u_g$ is the slip ratio, $T_r = T_p/T_g$ the temperature ratio, and b the velocity of sound in the gas/particle suspension.

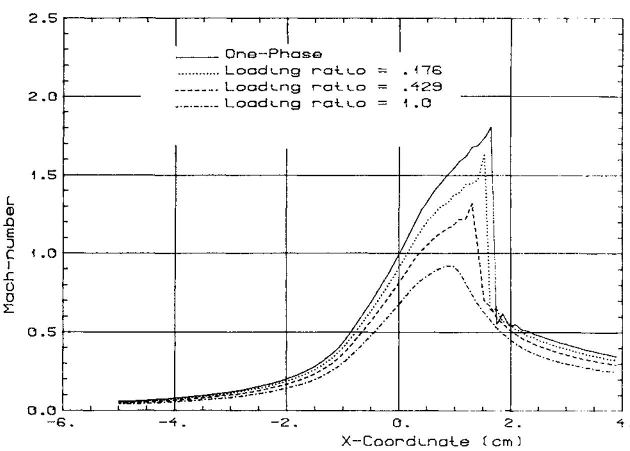

Fig. 2 Gas phase Mach number distribution (particle diameter of 2 μm).

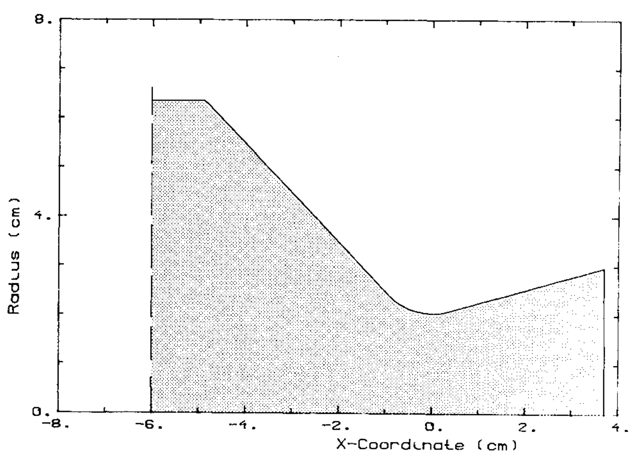

Fig. 1 JPL nozzle contour.

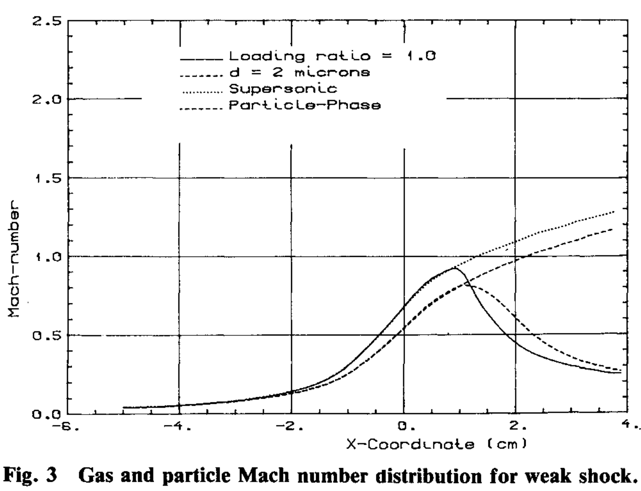

Fig. 3 Gas and particle Mach number distribution for weak shock.

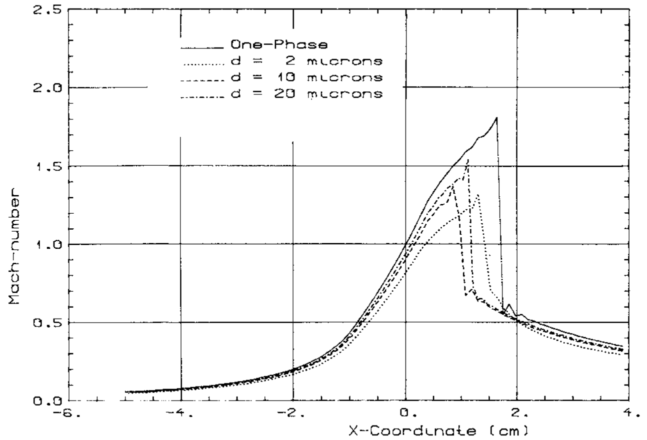

Fig. 4 Gas Mach number distribution (loading ratio of 0.429).

In order to calculate the critical velocity of sound in the suspension, the slip ratio is

$$S = 1 - w/b \tag{17}$$

where $w = u_g - u_p$.

For quasi-gas calculation, the critical condition occurs at the minimum cross-sectional area. The quasi-gas Mach number M_Q is

$$M_Q = M_g (c/b) \tag{18}$$

Using Eqs. (16-18) with the condition in Fig. 3 gives the quasi-gas Mach number $M_Q = 1$, i.e., shocked flow in the nozzle. Furthermore, this implies that the gas/particle mixture can have a region with supersonic (quasi-gas supersonic) flow and, in order to satisfy a given back pressure, a mixture-shock will occur.

Figure 4 shows the Mach number distribution in the nozzle for loading ratio 0.429 and particle diam 2, 10, and 20 μm. The figure shows that the strength of the frozen shock increases with increased particle diameters. The location of the shock has a more complicated pattern. At first the shock location moves upstream and then downstream for increased particle diameters. This can be explained using physics. The smaller particles have more influence on the gas flow for the same particle loading. This is due to the fact that, for the same particle loading, the total particle surface area effective for momentum and energy exchange between gas and particles is greater in a two-phase flowfield involving smaller diameter particles. The diameter of the particles will then have influence on the flow upstream of the shock, the frozen shock, and the relaxation zone behind the frozen shock. In order to satisfy a given back pressure, the location of the frozen shock will move upstream and downstream dependent of the upstream condition, the pressure rise in the frozen shock, in the relaxation zone, and due to the cross-sectional area change.

Conclusions

Quasi-one-dimensional gas/particle flows with shock inside a nozzle have been studied numerically herein. The results show: 1) the complexity of the shock pattern is dependent on loading ratio and particle diameter and 2) the strength of the shock is influenced stongly by the presence of the particles. The study also shows the mixture-shock condition in the gas/particle nozzle flow, caused by the fact that the velocity of sound for the gas/particle mixture is less than the velocity of sound for the gas phase, and the quasi-gas Mach number is above one.

References

[1]Rudinger, G., "Flow of Solid Particles in Gases," AGARDograph No. 222, 1976.

[2]Rudinger, G., "Some Effects on Finite Particle Volume on the Dynamics of Gas-Particle Mixtures," *AIAA Journal*, Vol. 3, July 1965, pp. 1217-1222.

[3]Kriebel, A. R., "Analysis of Normal Shock Waves in Particle Laden Gas," *Transactions of ASME, Journal of Basic Engineering*, Dec. 1964, pp. 655-665.

[4]Zucrow and Hoffmann, *Gas Dynamics*, Vol. II, John Wiley & Sons, Inc., New York, 1977, pp. 53-67.

[5]Wallis, G. B., "One-Dimensional Two-Phase Flow," McGraw-Hill Book Co., New York, 1969.

[6]Di Giacinto, M., Sabetta, F., and Piva, R., "Two-Way Coupling Effects in Dilute Gas-Particle Flows," *Transactions of ASME Journal of Fluids Engineering*, Vol. 104, Sept. 1982, pp. 304-312.

[7]Chang, I.-S., "One- and Two-Phase Nozzle Flows," *AIAA Journal*, Vol. 18, Dec. 1980, pp. 1455-1461.

[8]Sharma, M. P. and Crowe, C. T., "A Novel Physico-Computational Model of Quasi One-Dimensional Gas-Particle Flows," *Transactions of ASME, Journal of Fluids Engineering*, Vol. 100, Sept. 1978, pp. 343-349.

[9]Chang, I.-S., "Three-Dimensional, Two-Phase Supersonic Nozzle Flows," *AIAA Journal*, Vol. 21, May 1983, pp. 671-678.

[10]Rudinger, G., "Some Properties of Shock Relaxation in Gas Flows Carrying Small Particles," *The Physics of Fluids*, Vol. 7, May 1964, pp. 658-663.

[11]Carlson, D. J. and Høglund, R. F., "Particle Drag and Heat Transfer in Rocket Nozzles," *AIAA Journal*, Vol. 2, Nov. 1964, pp. 1980-1984.

[12]Cuffel, R. F., Back, L. H., and Massier, P. F., "Transonic Flowfield in a Supersonic Nozzle with Small Throat Radius of Curvature," *AIAA Journal*, Vol. 7, July 1968, pp. 1364-1366.

[13]Førde, M., "Velocity of Sound in Non-Equilibrium Gas/Particle Flows," Division of Aero- and Gas-Dynamics, NTH Trondheim, Norway, Rept. IFAG-A-159, 1984.

Surface Renewal Model for Turbulent Boundary-Layer Flow

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Introduction

THE standard approach to characterizing the wall region for turbulent boundary-layer flow involves the use of the van Driest[1] damping factor relationship for mixing length ℓ

$$\ell^+ = \kappa y^+ [1 - \exp(-y^+/a^+)] \tag{1}$$

where κ is an empirical constant and a^+ is associated with the dimensionless frequency of idealized fluid oscillations near the wall; κ is generally set equal to 0.41 and a^+ is specified by empirical correlations. However, the damping factor approach has not been found to provide a secure basis for generalization. For example, the damping factor approach has been of limited value in characterizing transitional turbulent flow and variable property flows. Furthermore, damping factor formulations for eddy thermal conductivity (or turbulent Prandtl number, Pr_t) appear to have

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